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Calculation of areas and static balance of barrel vaults with lunettes

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— Abstract—

Masonry structures were conceived from robust elements whose essential function is to support compressive forces. The dimensions of these constructions arose from principles of statics and geometry. Therefore, in this work a purely geometric and balanced approach is proposed to determine the forces that a vault with lunettes must support. To avoid the use of sophisticated programs, new equations are developed to calculate the surface that is generated and, from it, calculate the respective volume and weight.

The proposed equations are checked for the case of a groin vault and compared with a particular applied mathematics solution. After that, they are used in a reconstruction project in a small temple where materials from the region are used. The capacity of the material and the structural element as a whole is checked through balance conditions.

Keywords:

Vault with lunettes; groin vault; static equilibrium.

BACKGROUND

In old constructions, the use of the arch represents one of the most used structural elements to cover small to large spaces. From the tenth-century straight barrel vaults were built without any opening, and later windows were added, as in figure 1a) and groin vaults as in figure 1b). Arch ribs represent the transverse support and allow the vault to be divided into short segments. The symbiosis of the barrel vault with ports and the groin vault can be accepted as the basis for the construction of the barrel vault with lunettes, where the ports extend perpendicularly as an intersection of two-barrel vaults.

a) Barrel vault with ports **b)** Groin vaults

Figure 1. Example of two types of vaults (Laule & Geese, 2003)

The structural design of these massive structures was based on principles of statics and geometry, and among the firm defenders of this approach is Santiago Huerta (2004). In addition, finite element models require the adoption of physical hypotheses of a little-known material (Meli, 1998), which motivates the development of this analysis.

INTRODUCTION

Barrel vaults with lunettes, usually with semicircular arches, are formed by the perpendicular intersection of two of them (Figure 2). To correspond to a vault with lunettes, the diameters between one and the other vault are different, or at least, they must be at different heights. When the diameters are equal and start from the same level you can get a groin vault.

Figure 2 shows a model with two vaults of different diameters starting from the same level. Figure 3 shows the most common case, where the lunettes have a small length that extends to the cloth of the walls as in figure 4, which corresponds to the temple of San Pedro and San Pablo in Ecatzingo, State of Mexico. These are lunettes similar to those in Figure 3. Note that the extreme arch of the lunettes has a string of dimensions less than its diameter, a case that will be studied in this work. In that temple, the thin vault next to the dome collapsed. Much of the side wall, façade, and bell tower were also lost.

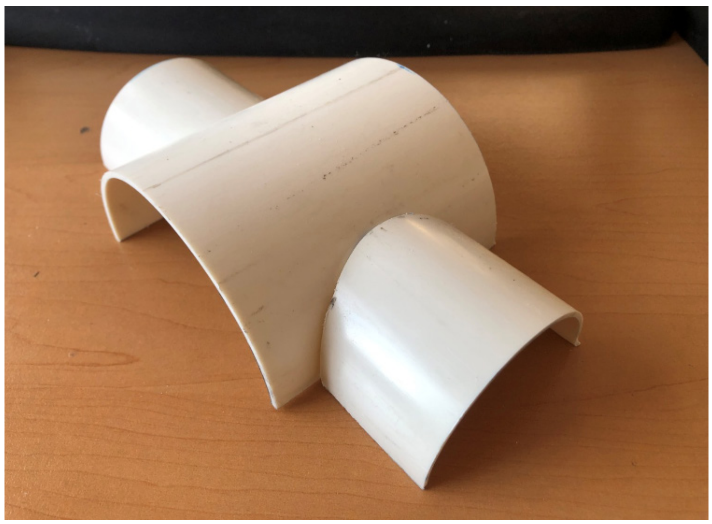

Figure 2. Model barrel vault with lunettes. Source: Own elaboration

Figure 3. Model barrel vault with cropped lunettes. Source: Own elaboration

Figure 4. Vault with lunettes. Source: Elaboration INAH (2018)

OBJECTIVE

This paper presents equations to calculate the volume of these two types of vaults: lunettes and groin. The equations are analytically checked for the second case because it is a direct comparison. After that, they are applied in a particular way to a current restoration project.

MATERIALS AND METHODS

Calculation of areas

To develop the equations, some geometric considerations are required, such as those given in Figures 5 and 6. The first one is a lunette with its beginning from the diameter but at a different height h, while in 5, in addition to the previous condition, the beginning of the lunette can be even higher h''. The vertical area differential dA allows to obtain the area of the lunette, and the horizontal differential dA' allows to obtain the area that is eliminated from the barrel vault.

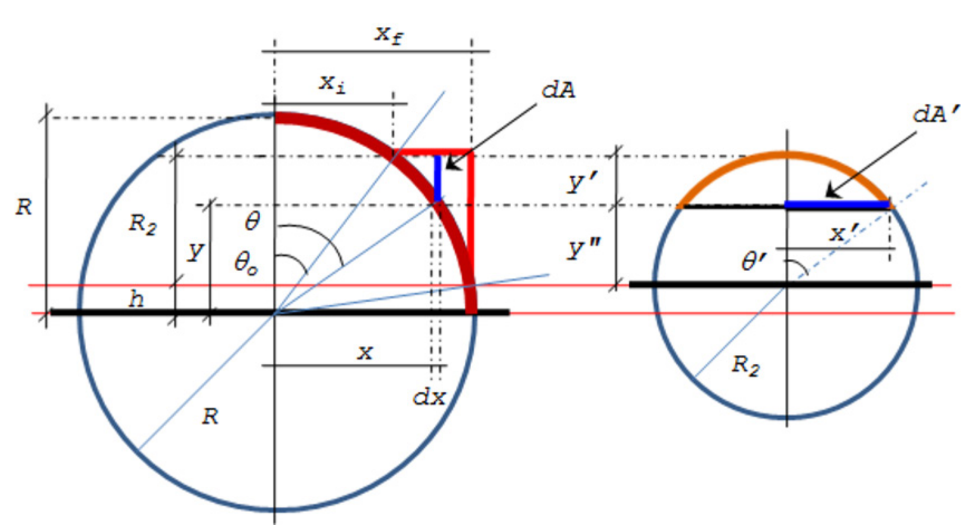

Figure 5. Diameters at different heights *h*. Source: Own elaboration

The base geometry usually includes the rope and arrow of the barrel vault and lunette. This string does not coincide, on many occasions, with the diameter as seen in Figure 4, so it is necessary to calculate the radius for both cases, this is R and R2 with the general equation (1).

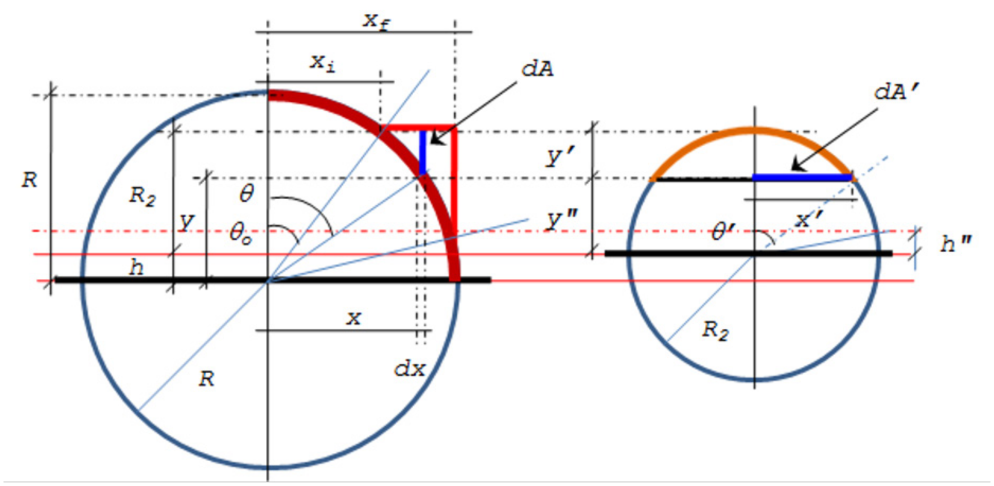

Figure 6. Lunette beginning on its diameter *h*". Source: Own elaboration

$$R = \frac{x^2 + z^2}{2z} \tag{1}$$

where *x* is half rope and *z* the arrow. According to Figure 4, *x*' and *y*' would be *x* and *z* in equation (1) respectively. In figures 5 and 6, the circles on the left correspond to the barrel vault and those on the right to the lunette. In the case of the edge vault, in both directions, they are vaults or lunettes, whatever the name preference could be.

To calculate the surface of the lunette, the area differential $dA= l'\cdot dx$ will be integrated, while to obtain the area that is extracted from the vault the differential $dA'= x'\cdot dy$ will be integrated. Note that different differentials are used for each case. dA is made dependent on x, and dA' is made dependent on y.

To calculate the arc l' of the lunette that is formed at the distance x (left circle) you will get x' (right circle) and thus obtain the angle $\boldsymbol{\theta}'$. Since we know the radius, whether by data or calculated with the equation (1), the length depends on the angle $\boldsymbol{\theta}'$.

$$l' = R_2 \cdot \frac{\pi \cdot \theta'}{90} \tag{2}$$

From previous figures,

$$y = \sqrt{R^2 - x^2} \tag{3}$$

$$x' = \sqrt{R_2^2 - y''^2} \tag{4}$$

$$y'' = R_2 - y' \tag{5}$$

$$y' = (R_2 + h) - y \tag{6}$$

$$x' = \sqrt{R_2^2 - \left(\sqrt{R^2 - x^2} - h\right)^2} \tag{7}$$

So we can express l' in terms of x as,

$$l' = \frac{\pi \cdot R_2}{90} \cdot sen^{-1}\left[\frac{\sqrt{R_2^2 - \left(\sqrt{R^2 - x^2} - h\right)^2}}{R_2}\right] \tag{8}$$

Using equations (6) and (1), we can rewrite x' as,

$$x' = \sqrt{2R_2(y') - y'^2} = \sqrt{R_2^2 - h^2 + y(2h - y)} \tag{9}$$

Due to the above, we have:

$$A_L = \frac{\pi \cdot R_2}{90} \int_{x_i}^{x_f} sen^{-1}\left[\frac{\sqrt{R_2^2 - \left(\sqrt{R^2 - x^2} - h\right)^2}}{R_2}\right] dx \tag{10}$$

$$A_h = \frac{\pi \cdot R_2}{90} \int_{y_i}^{y_f} sen^{-1}\left[\frac{\sqrt{R_2^2 - h^2 + y(2h - y)}}{R_2}\right] dy \tag{11}$$

For the correct use of the integrals, it is necessary to define the limits x_i and x_f so that for the first x_i'= 0 and for the second x_f'= rope at height h" from where the lunette starts. If the axes of the diameters are at the same height and in addition, the lunette starts from those axes, h=h"=0 and x_i'=0, x_f'=R_2. For this,

$$x_i = \sqrt{R^2 - (R_2 + h)^2} \quad (12)$$

$$x_f = \sqrt{R^2 - (h + h")^2} \quad (13)$$

For limits y_i and y_f we have:

$$y_i = h" \quad (14)$$

$$y_f = R_2 \quad (15)$$

FIRST VALIDATION

To verify the results of equations (10) and (11), take a groin vault as shown in Figure 7. A_L is the area of the lunette to be calculated, A_h is the area to be eliminated by the intersection of the vaults and A_B is the complete area of the vault in a direction without lunettes. For this case R=R_2 =5.25m and h=h" =0m.

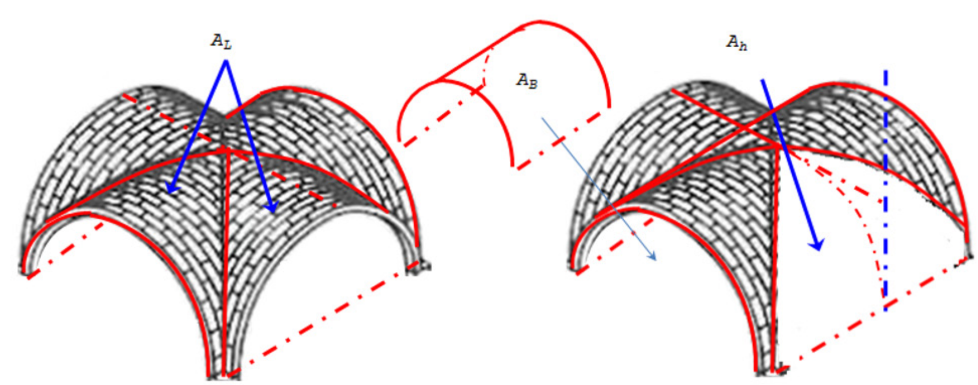

Figure 7. Groin vault. Source: Mixed elaboration with Google (2021)

From equations (12) and (13), x_i=0.0m, x_f=5.25m and from equations (14) and (15) y_i=0, y_f=5.25m. As proof, from equation (7), it should be verified that x_i'=0 y x_f'=R_2. Now from equations (10) and (11):

$$A_L = \frac{\pi \cdot R_2}{90} \int_0^{5.25} sen^{-1}\left[\frac{\sqrt{27.5625 - \left(\sqrt{27.5625 - x^2}\right)^2}}{5.25} \right] dx$$

$$A_h = \frac{\pi \cdot R_2}{90}\int_0^{5.25} sen^{-1}\left[\frac{\sqrt{27.5625 - y^2}}{5.25}\right]dy$$

That is, A_L=31.46514751456868m², A_h=55.125m².

Verification

To test the above results, the area of the vault A_B will be calculated until the end of the lunette (figure 7), that is, halfway through the vault, that is:

$$A_B = \pi \cdot R \cdot (l) = \pi \cdot R^2 = 86.59014751456868m^2$$

If we subtract the area A_h to A_B, the lunette area A_L must result, thus:

$$A_L = A_B - A_h \tag{16}$$

$$A_L = 86.59014751 - 55.125 = 31.46514751456868m^2$$

The remarkable thing about equations (10) and (11) is that they show no difference in the calculation of A_L with equation (16). Another check of the calculation of A_L can be found in Salinas and Costa (2017), where ½·A_L for λ=5.25 is obtained by:

$$A_L = 2 \cdot \lambda^2\left(\frac{\pi}{2} - 1\right) = 31.46514751456868m^2 \tag{17}$$

Again a result is obtained with a negligible difference concerning equation (10), which in turn confirms the validity of equation (11). The number of decimals has been exaggerated to prove the accuracy of the proposed equations.

Second validation

A second exercise is done for a new radius R=3.6m. As a groin vault will continue to be considered, the diameters for both vaults will be the same, and the lunettes start from the same level. Therefore, $R=R_2$=3.6m, h=h”=0m, so from equations (12) to (15) $x_i=y_i$=0, $x_f=y_f$=3.6m. Returning to equations (10) and (11), we have,

$$A_L = \frac{\pi \cdot R_2}{90}\int_0^{3.6} sen^{-1}\left[\frac{\sqrt{12.96 - \left(\sqrt{12.96 - x^2}\right)^2}}{3.6}\right]dx$$

$$A_h = \frac{\pi \cdot R_2}{90} \int_0^{3.6} sen^{-1}\left[\frac{\sqrt{12.96 - y^2}}{3.6}\right] dy$$

So, A_L=14.7950407906m², A_h=25.92m².

Verification

The area of the vault is:

$$A_B = \pi \cdot R \cdot (l) = \pi \cdot R^2 = 40.71504079 m^2$$

Recalculating the lunette area with the difference of the vault area minus the hollow area (equation 16), we have,

$$A_L = A_B - A_h = 14.7950407906 m^2$$

It can be seen that the value of A_L matches with that calculated with equation (10). If equation (17) is applied again, now with λ=3.6m, we have:

$$A_L = 2 \cdot \lambda^2 \left(\frac{\pi}{2} - 1\right) = 14.7950407905 m^2$$

Then, equation (10) or equation (16) gives the same result as equation (17) taken as a reference. It is important to clarify that equation (17) is only applicable to complete lunettes that start from the diameter of their end arc. In this work, the equations offer solutions to groin vaults or vaults with lunettes, as shown in Figure 2.

RESULTS AND DISCUSSION

Case Study

Once the proposed equations have been reviewed, they will be applied to the vault of the San Juan temple in Malinalco, State of Mexico. Figure 8 shows the arches covered with canvas, which support the orthogonal vaults. The height of the arches of the main nave is slightly higher than that of the side arches. This is shown in Figure 9. The figure on the left depicts the barrel vault, which goes in the longitudinal direction, and to the right, it shows the lunette, that is, the side view of the temple.

Figure 8. Temple of San Juan, Malinalco, State of Mexico. Source: Own elaboration

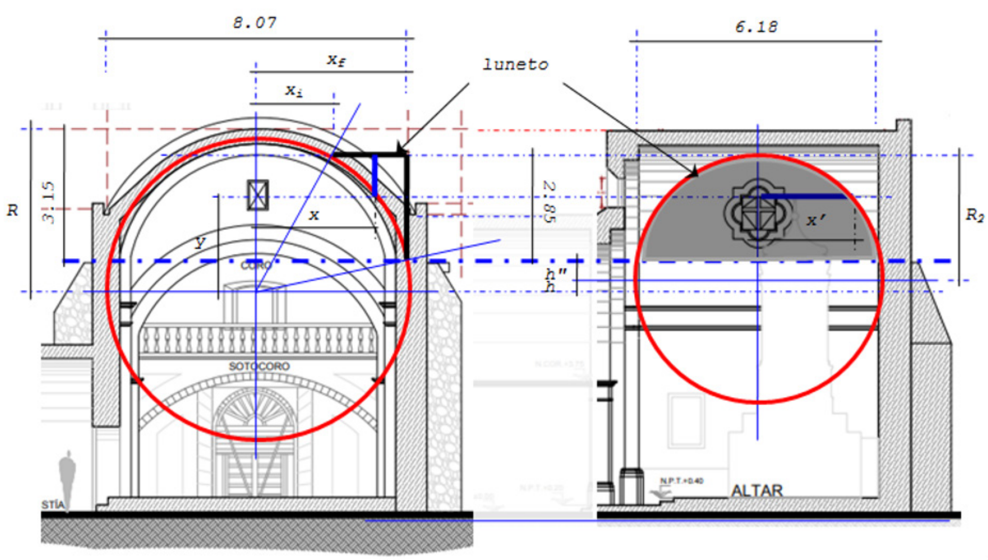

Figure 9. Barrel and lunette vault. Source: Mixed elaboration with INAH (2020)

For the project data, we found R=4.159m and R_2=3.1m. In addition h=0.759m and h"=0.25m. All the above values were adjusted as best as possible to the available topographical surveys and tried to avoid inconsistencies. Note that the lunette does not start from its diameter. This requires the lunette to be integrated from a height greater than its diameter and consequently, x_f also does not cover the entire radius R. Therefore, now it is not possible, as verification, to obtain the lunette area by subtracting the hollow areas from the vault area since that is only valid for groin vaults. Only three decimal places will be used from here on since for a real case it is inappropriate to keep more than three decimal places.

The limits for integrals are:

$$x_i = \sqrt{4.159^2 - (3.859)^2} = 1.551m$$

$$x_f = \sqrt{4.159^2 - (1.009)^2} = 4.0136m$$

and the limits y_i and y_f are:

$$y_i = 1.009m$$

$$y_f = 3.859m$$

Lunette area A_L and intersection area A_h,

$$A_L = \frac{\pi \cdot R_2}{90} \int_{1.551}^{4.0136} sen^{-1}\left[\frac{\sqrt{9.61 - \left(\sqrt{17.297 - x^2}\right)^2}}{3.1}\right] dx$$

$$A_h = \frac{\pi \cdot R_2}{90} \int_{1.009}^{3.859} sen^{-1}\left[\frac{\sqrt{9.033919 + y(1.518 - y)}}{3.1}\right] dy$$

Where A_L = 11.237m² and A_h =16.848m². The area of the barrel vault with the gaps caused by the intersection of the lunettes is:

$$A_B - A_h = l_B(l) - A_h \tag{18}$$

Where l_B is the arch of the vault obtained with equation (19) and the angle with equation (20) from *h*" (figure 9).

$$l_B = \frac{\pi \cdot R}{90}\theta \tag{19}$$

and

$$\theta = tg^{-1}\left(\frac{x_f}{(h + h")}\right) = tg^{-1}\left(\frac{4.0136}{1.009}\right) \tag{20}$$

The length of the vault *l* is 6.18m (figure 9), so that A_B=68.086m², so that from equation (18), A_B-A_h=34.39m². The total area of the vault with lunettes is 56.864m².

Static Balance

If a barrel vault segment or dome cap where its weight is known is taken, the vertical reaction at its edges should be equal to the weight of that section (Figure 10).

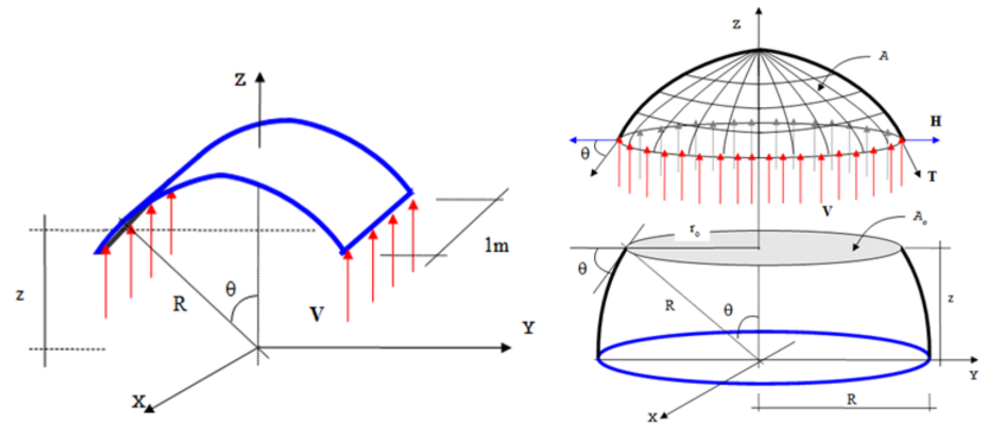

Figure 10. Vault and dome segments in balance. Source: Miramontes, 2016

To determine the weight of any of the sections shown in Figure 10, it is necessary to know their geometry, which indicates that in any cut the tangent can be determined at any point at the height *z*. The arch length for the vault is a trivial topic, so it may be of greater interest to include the equations for obtaining the cylindrical cap area in Figure 10.

$$A = 2\pi R^2\left(1-\cos\left(\frac{\pi\theta^o}{180^o}\right)\right) \quad (21)$$

$$\theta = \cos^{-1}\left(\frac{z}{R}\right) \quad (22)$$

Knowing area *A*, the volume is obtained by multiplying it by thickness *t*, and in turn, the weight *W* is obtained by multiplying it by the volumetric weight of the material γ_m (equation 23),

$$W = A \cdot t \cdot \gamma_m \quad (23)$$

The vertical reaction *V* is obtained by dividing the weight by the support length, which is obtained directly from the cut section. With this, the tangential force *T* and the horizontal force *H* can be obtained for any of the above cases:

$$T = \frac{V}{\operatorname{sen}\theta} \tag{24}$$

$$H = T\cos\theta \tag{25}$$

This principle applies to the section to be analyzed in the San Juan temple. Once the area of the vault was determined, including the lunettes, a thickness of 0.2m was proposed with local tezontle (Miramontes, 2021). For this material, a specific weight of 1.735T/m³ was obtained, so its total weight is 19.732Ton. Three samples used to calculate the weight of the material are included in Figure 11, and the results for dry weight and wet weight are shown in Table 1. In addition to its own weight, the vault receives an additional load due to the coating and finishing materials. Table 2 describes the weights per square meter to be considered in the vault analysis.

To evaluate the support offered by the lunette to the rest of the vault, the weight of the isolated lunette must be obtained, and its centroid calculated using the Varignon theorem. For this, it is necessary to calculate the static moment given by:

$$\dot{X} \cdot A_L = \frac{\pi \cdot R_2}{90} \int_{x_i}^{x_f} x \cdot sen^{-1}\left[\frac{\sqrt{R_2^2 - \left(\sqrt{R^2 - x^2} - h\right)^2}}{R_2}\right] dx \tag{24}$$

The result of equation (26) is divided by the area of the lunette, and distance $\dot{X}$ is obtained. To obtain $\dot{x}$, we simply take the difference with the barrel vault's radius. It is important to add that the lunette is not the only one that offers support to the vault since an arc function is also generated in the area A_h (see figure 7).

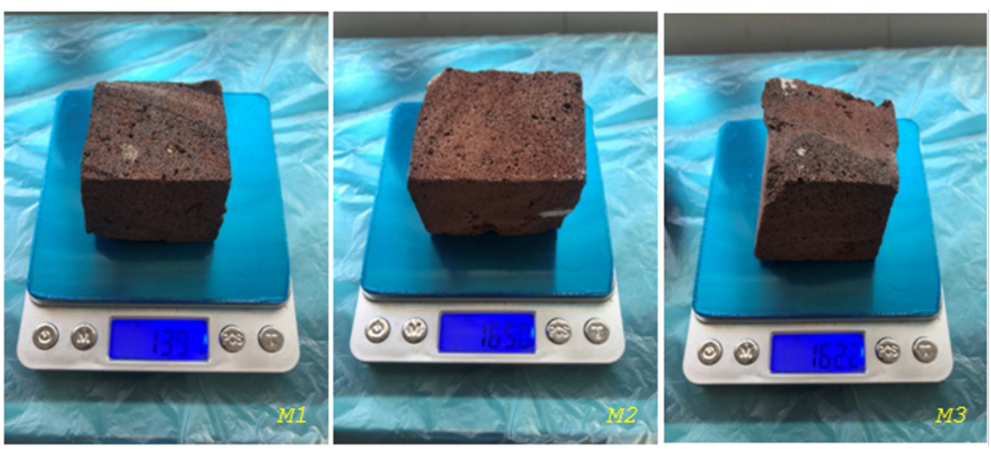

Figure 11. Dry weight of samples. Source: Own elaboration

Table 1
Vault's materials volumetric weights

SAMPLE	Volume cm^3	W_s gr	W_h gr	γ_s Ton/m^3	γ_h Ton/m^3	% empy
M1	84	139.1	145.7	1.655	1.735	7.857
M2	113	165.0	174.1	1.460	1.541	8.054
M3	114	162.2	171.3	1.423	1.506	7.982

Source: Own elaboration

Table 2
Weight per m^2 of other materials

SURFACE	MATERIAL	WEIGHT kg/m^2
	Mortar	55.5
EXTRADOS	Tiles	54.0
	Waterproofing	2.0
INTRADOS	Mortar (or stucco)	37
TOTAL		**148.5**

Source: Own elaboration

Of (21) results $\dot{X}A_L$=35.113m^3, which $\dot{X}$=3.125m. This indicates that the results or weight of the lunette plus the weight of the materials is at $\dot{X}$=0.89m from its end. If the total weight is,

$$W_L = A \cdot t \cdot \gamma_t + A \cdot W_m = 11.237(0.2)(1.735) + 11.237(0.149) = 3.899 + 1.669 = 5.568\,Ton$$

a horizontal force H_L = 1.599Ton is generated at the highest point caused by the rotation of the lunette. Figure 11 shows a model where the distances of the resulting W_L to the center of the vault and the end of the lunette are linked.

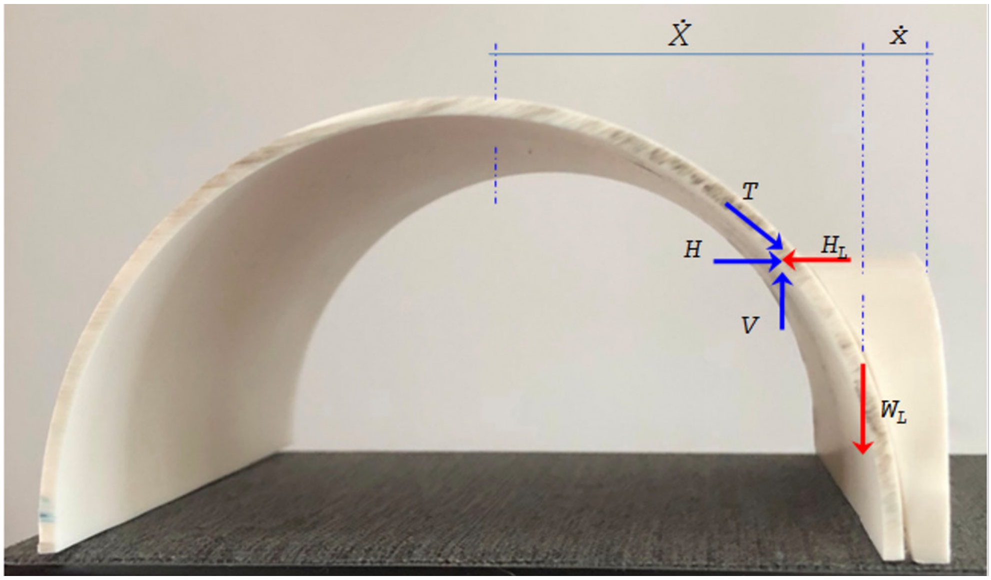

Figure 12. Model vault with lunette. Source: Own elaboration

From Figure 8, *V*, *T*, and *H* values are obtained for the arc that is formed at the highest point of the lunette, so that with equations (22) to (25), we have: θ=21.895°, *W*=1.575Ton, *T*=2.112Ton and *H*=1.960Ton. For the lowest point of the lunette, we have θ=75.96°, *W*=5.465Ton, *T*=2.816Ton and *H*=0.683Ton, and *H*=0.683Ton. The average of *H*=1.322Ton is lower than the H_L total of the lunette.

Note that the compression (tangential force *T*) increases as it descends in the arch of the vault while the horizontal force *H* decreases. For complete half-point arcs *H*=0. If you take a unit width of the vault and the maximum value of *T*, you can see that the compressive stress is very low, that is,

$$\sigma = \frac{T}{b \cdot t} = 1.408 kg / cm^2$$

σ is lower than the common compression capacity in stone masonry, which is very close to 20kg/cm² (2.0MPa) according to the Complementary Technical Standards for the Design and Construction of Masonry Structures or according to some experimental results (NTC, 2020; Mauritius, 2021).

CONCLUSIONS

We proposed new equations to calculate the area of vaults with lunettes and, with this value, calculate the weight and forces that are generated along its geometry. The equations were validated for the case of groin vaults by comparing them directly and indirectly, using an equation for complete lunettes, obtaining a negligible difference, and then applying it in a real case. Therefore, the main objective of this work is considered achieved.

Once the area of the vault is known, the weight of the vault is calculated, and employing basic principles of balance, the forces for a current reconstruction project in a temple in the State of Mexico are determined.

From the results obtained, it is concluded that the efforts by its own weight are lower than the nominal capacity of the material for a static case, which is per the concept and design for this type of structure.

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